The Steady Magnetic Field
Part 2

Prepared By

Dr. Eng. Sherif Hekal

Assistant Professor – Electronics and Communications Engineering

Agenda

- Magnetic Flux And Magnetic Flux Density
- Maxwell’s Equations for Static Fields
- Gauss’s law for magnetic field
- Magnetic Boundary Conditions
- Magnetic Forces and Inductance
- Magnetic energy density

Magnetic Flux and Flux Density

We are already familiar with the concept of electric flux:

$$\Psi = \int_s \mathbf{D} \cdot d\mathbf{S} \quad \text{Coulombs} \qquad (1)$$

in which the electric flux density in free space is: $\mathbf{D} = \epsilon_0 \mathbf{E}$ $\mathrm{C/m^2}$

and where the free space permittivity is $\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F/m}$

In a similar way, we can define the magnetic flux in units of Webers (Wb):

$$\Phi = \int_s \mathbf{B} \cdot d\mathbf{S} \quad \text{Webers} \qquad (2)$$

in which the magnetic flux density (or magnetic induction) in free space is: $\mathbf{B} = \mu_0 \mathbf{H}$ $\mathrm{Wb/m^2}$ (3)

and where the free space permeability is $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H/m}$

This is a defined quantity, having to do with the definition of the ampere (we will explore this later).

Magnetic Flux and Flux Density

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

$$\Psi_{net} = \oint_s \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

If the same were to be done with magnetic flux density, we would find:

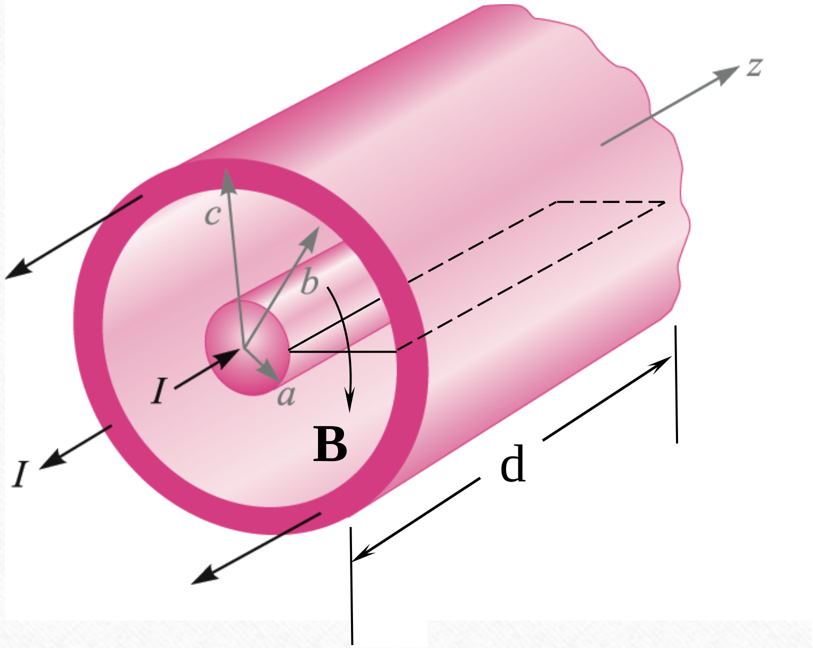
$$\Phi_{net} = \oint_s \mathbf{B} \cdot d\mathbf{S} = 0$$

The implication is that (for our purposes) there are no magnetic charges -- specifically, no point sources of magnetic field exist. A hint of this has already been observed, in that magnetic field lines always close on themselves.

Example: Magnetic Flux Within a Coaxial Line

Consider a length d of coax, as shown here. The magnetic field strength between conductors is:

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$



and so: $$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux is now the integral of **B** over the flat surface between radii a and b, and of length d along z:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

The result is: $$\Phi = \frac{\mu_0 I d}{2\pi} \ln\frac{b}{a}$$

The coax line thus "stores" this amount of magnetic flux in the region between conductors. This will have importance when we discuss inductance in a later lecture.

Maxwell’s Equations for Static Fields

We may rewrite the closed surface integral of **B** using the divergence theorem, in which the right hand integral is taken over the volume surrounded by the closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \; dv = 0$$

Because there is no isolated magnetic charge, the result is zero

$$\boxed{\nabla \cdot \mathbf{B} = 0} \qquad (4)$$

This result is known as **Gauss’ Law** for the magnetic field in point form.

Maxwell's Equations for Static Fields

It has just been demonstrated from Ampere' s Circuital Law that:

$$\oint H \cdot d\ell = I_{enclosed} = \int J \cdot ds$$

Applying the Stoke's theory

$$\therefore \oint H \cdot d\ell = \int (\nabla \times H) \cdot ds$$

$$= \int J \cdot ds$$

…..which is in fact one of Maxwell' s equations for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad (5)$$

This is Ampere' s Circuital Law in point form.

Maxwell's Equations for Static Fields

We already know that for a static electric field:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \qquad (6)$$

This means that: $\nabla \times \mathbf{E} = 0$ (applies to a static electric field)

Recall the condition for a conservative field: that is, its closed path integral is zero everywhere.

Therefore, a field is conservative if it has zero curl at all points over which the field is defined.

Maxwell's Equations for Static Fields

We have now completed the derivation of Maxwell's equations for no time variation. In point form, these are:

$$\nabla \cdot \mathbf{D} = \rho_v$$ Gauss' Law for the electric field

$$\nabla \times \mathbf{E} = 0$$ Conservative property of the static electric field

$$\nabla \times \mathbf{H} = \mathbf{J}$$ Ampere's Circuital Law

$$\nabla \cdot \mathbf{B} = 0$$ Gauss' Law for the Magnetic Field

where, in free space:

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

Significant changes in the above four equations will occur when the fields are allowed to vary with time, as we'll see later.

Maxwell's Equations in Large Scale Form

The divergence theorem and Stokes' theorem can be applied to the previous four point form equations to yield the integral form of Maxwell's equations for static fields:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv$$

Gauss' Law for the electric field

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

Conservative property of the static electric field

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

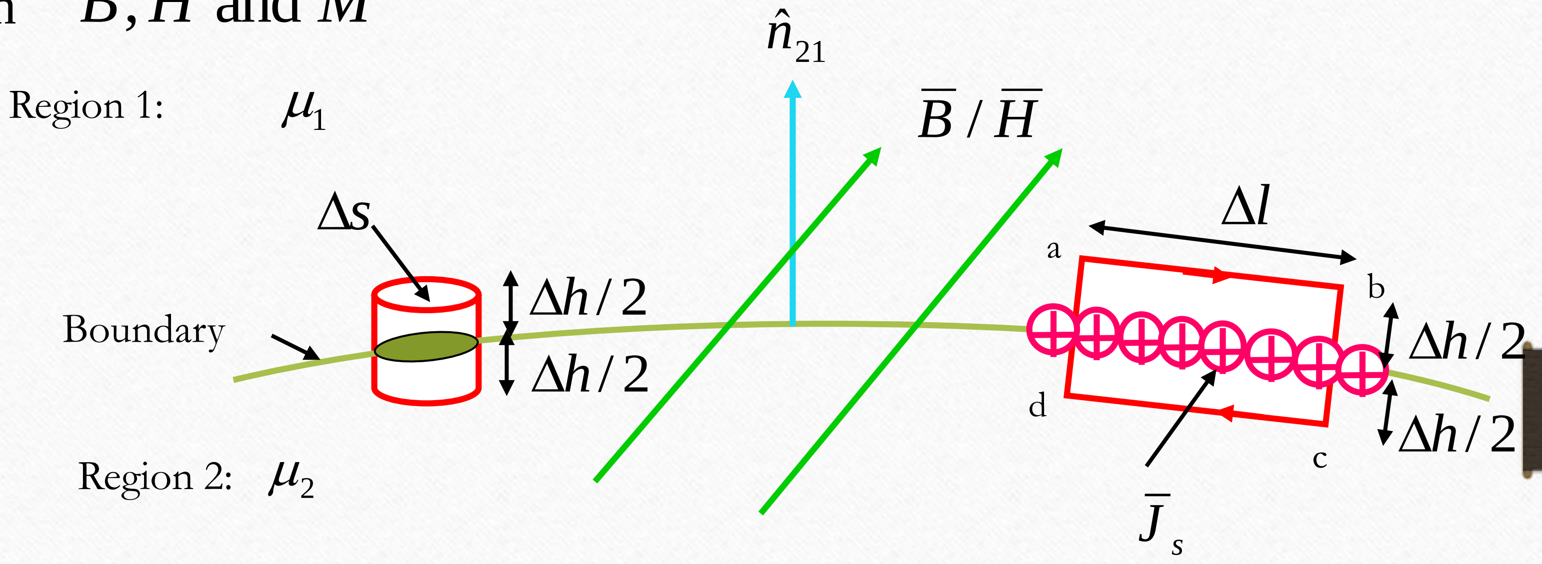
Ampere's Circuital Law

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' Law for the magnetic field

Magnetic Boundary Conditions

To find the relationship between $\overline{B}, \overline{H}$ and $\overline{M}$



To find normal component of $\overline{B}$ and $\overline{H}$ at the boundary

Consider a small cylinder as $\Delta h \to 0$ and use $\oint \overline{B} \cdot \overline{ds} = 0$

$$\oint \overline{B} \cdot \overline{ds} = B_{1n}\Delta s - B_{2n}\Delta s = 0$$

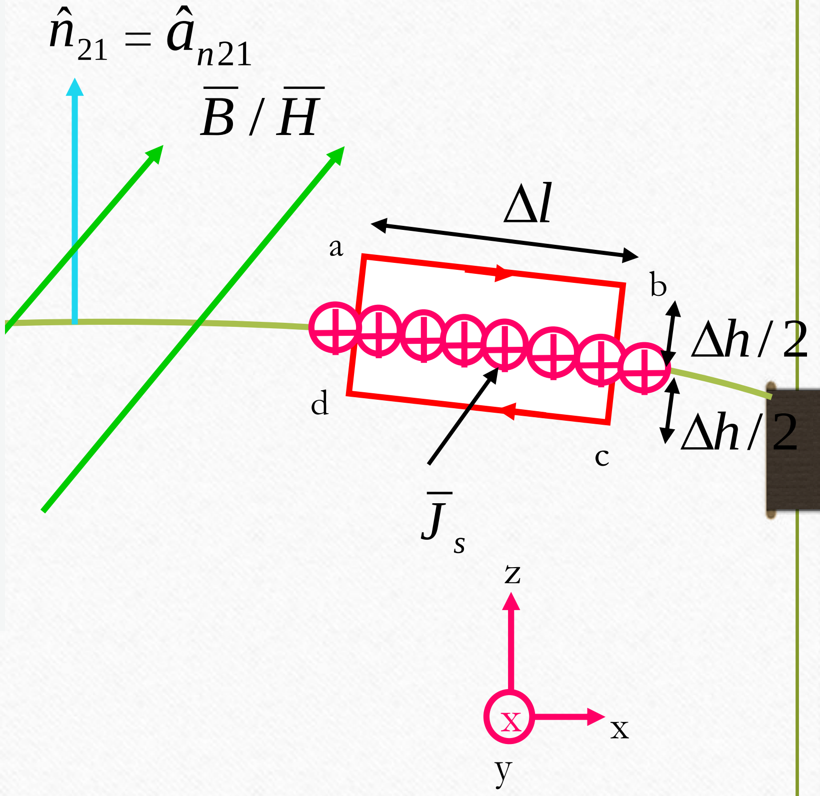
$$\therefore \quad B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n} \qquad (7)$$

Magnetic Boundary Conditions

To find tangential component of $\bar{B}$ and $\bar{H}$ at the boundary

Consider a closed abcd as $\Delta h \rightarrow 0$

and use $\oint_l \bar{H} \cdot \bar{dl} = I_{enc}$



$$H_{1t}\Delta l - H_{2t}\Delta l = I_{enc}$$
$$\therefore \quad H_{1t} - H_{2t} = K$$

where K is perpendicular to the directions of H_{1t} and H_{2t}

$$\hat{z} \times (\hat{x}) = \hat{y}$$

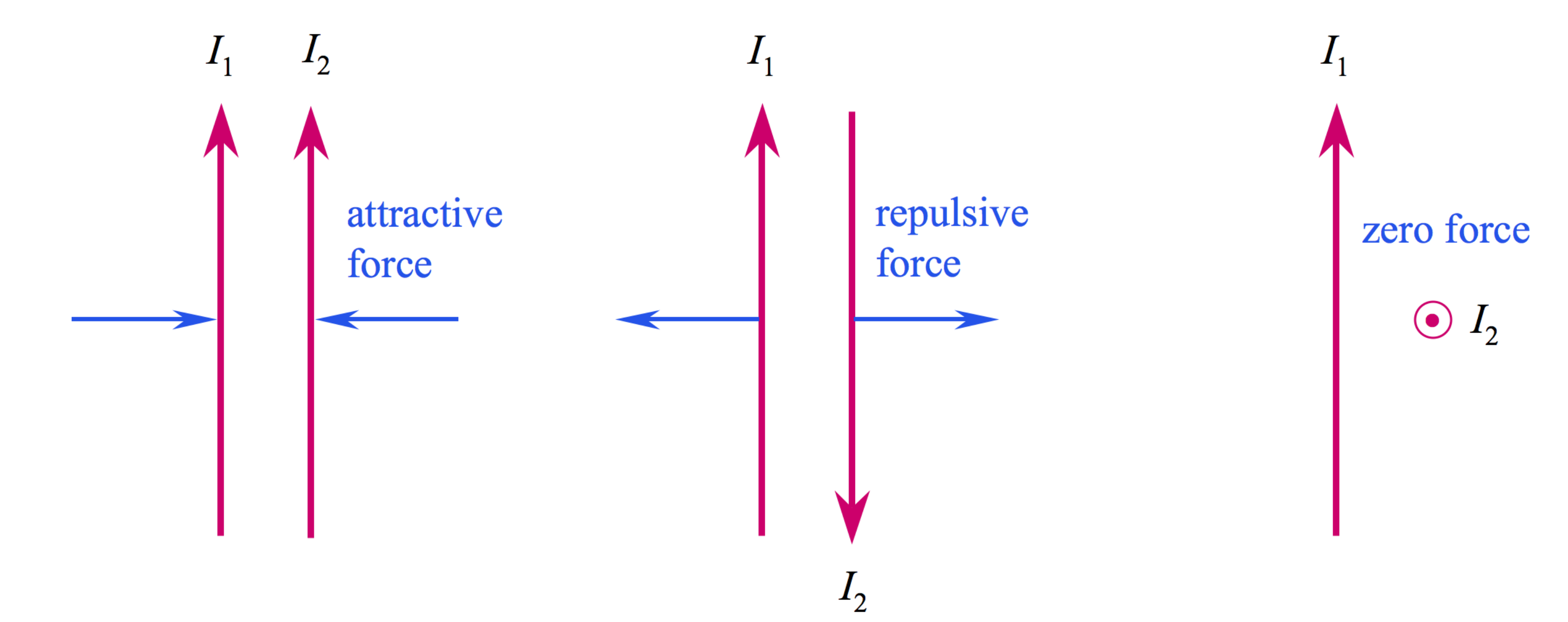
If $K = 0$:

$$\bar{H}_{1t} = \bar{H}_{2t} \quad \text{or} \quad \frac{\bar{B}_{1t}}{\mu_1} = \frac{\bar{B}_{2t}}{\mu_2} \qquad (8)$$

Motivating the Magnetic Field Concept: Forces Between Currents

Magnetic forces arise whenever we have charges in motion. Forces between current-carrying wires present familiar examples that we can use to determine what a magnetic force field should look like:

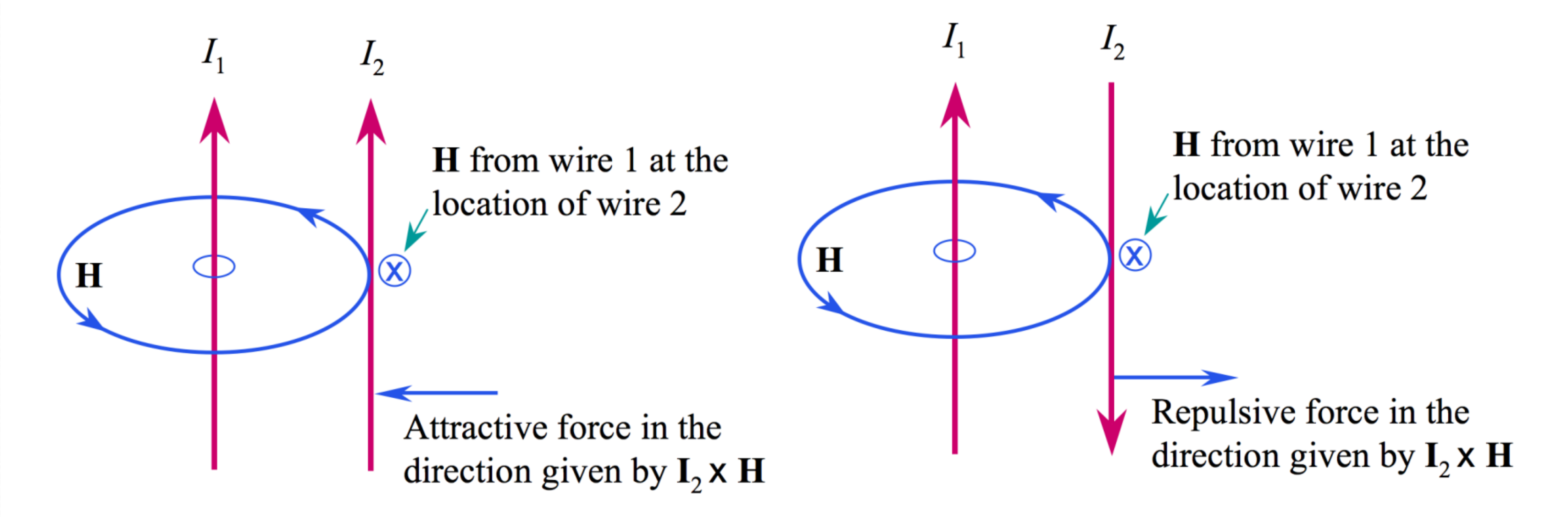
Here are the easily-observed facts:



How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

Magnetic Field

The geometry of the magnetic field is set up to correctly model forces between currents. The magnetic field intensity, **H**, circulates around its source, I_1, in a direction most easily determined by the right-hand rule: Right thumb in the direction of the current, fingers curl in the direction of **H**



Note that in the third case (perpendicular currents), I_2 is in the same direction as **H**, so that their cross product (and the resulting force) is zero. The actual force computation involves a different field quantity, **B**, which is related to **H** through **B** = μ_0**H** in free space. This will be taken up in a later lecture. Our immediate concern is how to find **H** from any given current distribution.

Force on a Moving Point Charge

In an electric field, the force on a charged particle is given by

$$\overline{F}_e = Q\overline{E} \qquad (9)$$

A charged particle in motion in a magnetic field of flux density B is found experimentally to experience a force whose magnitude is

$$\overline{F}_m = Q\overline{\mathrm{v}} \times \overline{B} \qquad (10)$$

Where v is the velocity, B is the flux density

The force on a moving particle arising from combined electric and magnetic fields is obtained easily by superposition,

$$\overline{F} = \overline{F}_e + \overline{F}_m \quad \text{or} \quad \overline{F} = Q\left(\overline{E} + \overline{\mathrm{v}} \times \overline{B}\right) \qquad (11)$$

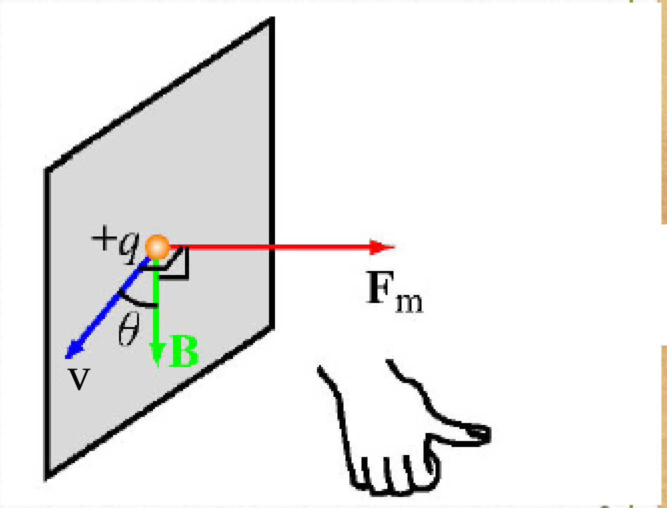
Also known as Lorentz force equation.

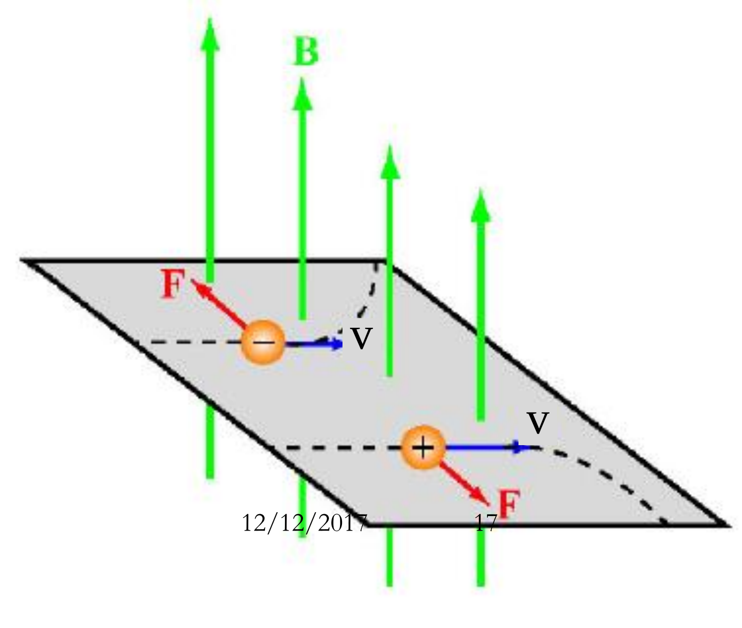
Force on a Moving Point Charge

Force on charge in the influence of fields:

Charge Condition	$\bar{E}$ Field	$\bar{B}$ Field	Combination $\bar{E}$ and $\bar{B}$
Stationary	$Q\bar{E}$	-	$Q\bar{E}$
Moving	$Q\bar{E}$	$Q\bar{v}\times\bar{B}$	$Q\left(\bar{E}+\bar{v}\times\bar{B}\right)$

- The electric force is usually in the direction of the electric field while, the magnetic force is perpendicular to the magnetic field
- The electric force acts on a charged particle whether or not it is moving, while the magnetic force acts moving charged particle only
- The electric force expends energy in displacing a charged particle, while the magnetic one does no work when the particle is displaced because it is perpendicular to the velocity





Force on a Moving Point Charge

D8. 4. The point charge $Q = 18$ nC has a velocity of 5×10^6 m/s in the direction $\mathbf{a}_v = 0.60\mathbf{a}_x + 0.75\mathbf{a}_y + 0.30\mathbf{a}_z$. Calculate the magnitude of the force exerted on the charge by the field: (*a*) $\mathbf{B} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ mT; (*b*) $\mathbf{E} = -3\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$ kV/m; (*c*) B and E acting together.

Ans. 660 μN; 140 μN; 670 μN

Force on a Differential Current Element

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$d\overline{F} = dQ\,\overline{\mathrm{v}} \times \overline{B}$$

$$\because dQ = \rho_v dv \quad \text{and} \quad \overline{J} = \rho_v \overline{\mathrm{v}}$$

$$\therefore d\overline{F} = \overline{J} \times \overline{B}\, dv$$

We saw in Chapter 8, part1 that $J\,\mathrm{d}v$ may be interpreted as a differential current element; that is,

$$\mathbf{J}\,dv = \mathbf{K}\,dS = I\,d\mathbf{L}$$

and thus the Lorentz force equation may be applied to a differential current filament,

$$\therefore d\overline{F} = Id\overline{\ell} \times \overline{B}$$

Force on a Differential Current Element

$$\therefore \overline{F} = \oint I d\overline{\ell} \times \overline{B}$$

$$= -I \oint \overline{B} \times d\overline{\ell} \qquad (12)$$

One simple result is obtained by applying (12) to a straight conductor in a uniform magnetic field,

$$\overline{F} = I\overline{L} \times \overline{B}$$

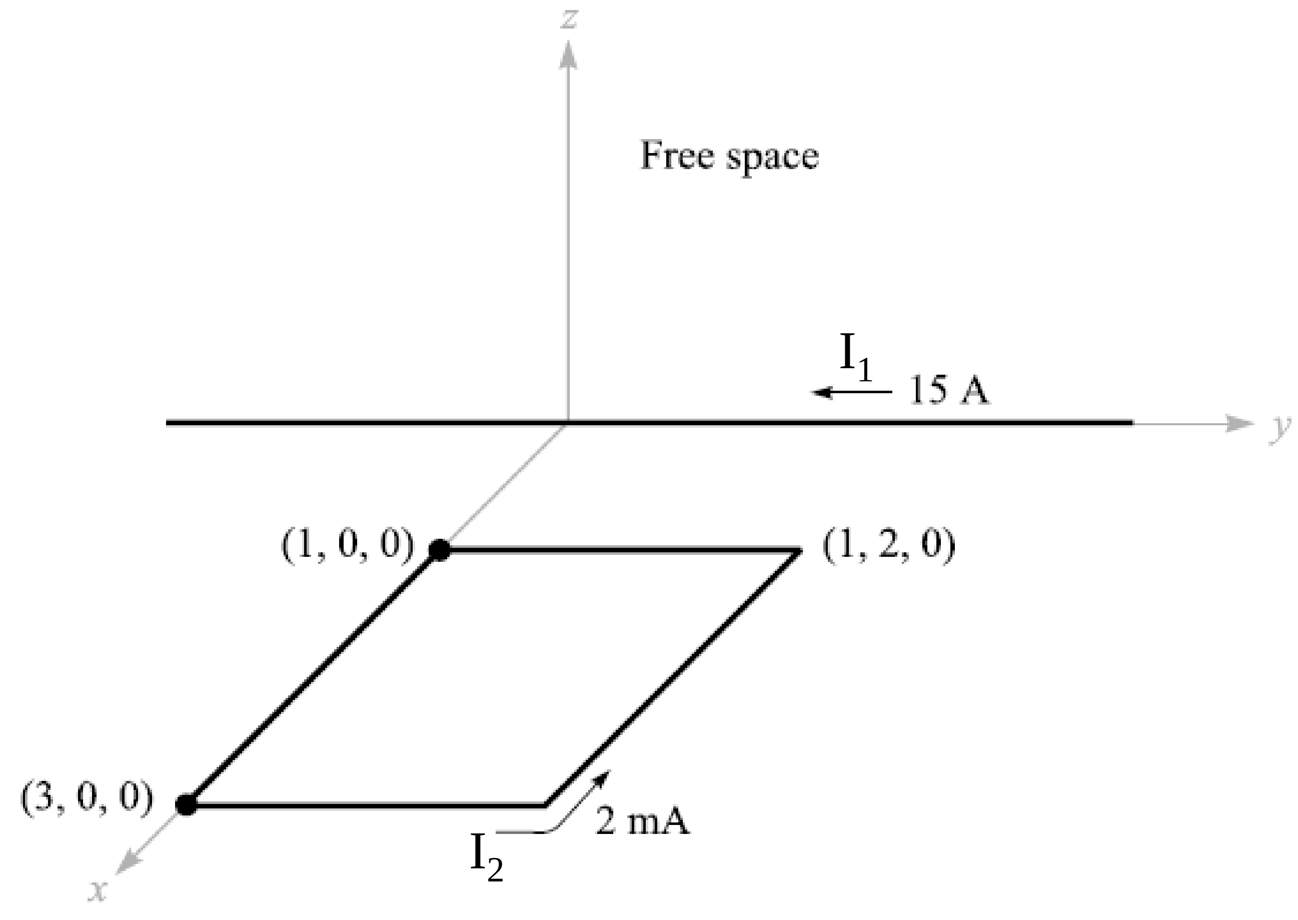
The magnitude of the force is given by the familiar equation

$$F = BIL\sin\theta \qquad (13)$$

where θ is the angle between the vectors representing the direction of the current flow and the direction of the magnetic flux density.

Force on a Differential Current Element

Ex. 8.3: A square conductor current loop is located in z = 0 plane with the edges given by the coordinates (1,0,0), (1,2,0), (3,0,0) and (3,2,0) carrying a current of 2 mA in anti clockwise direction. A filamentary current carrying conductor of infinite length along the y axis carrying a current of 15 A in the –y direction. Find the force on the square loop.



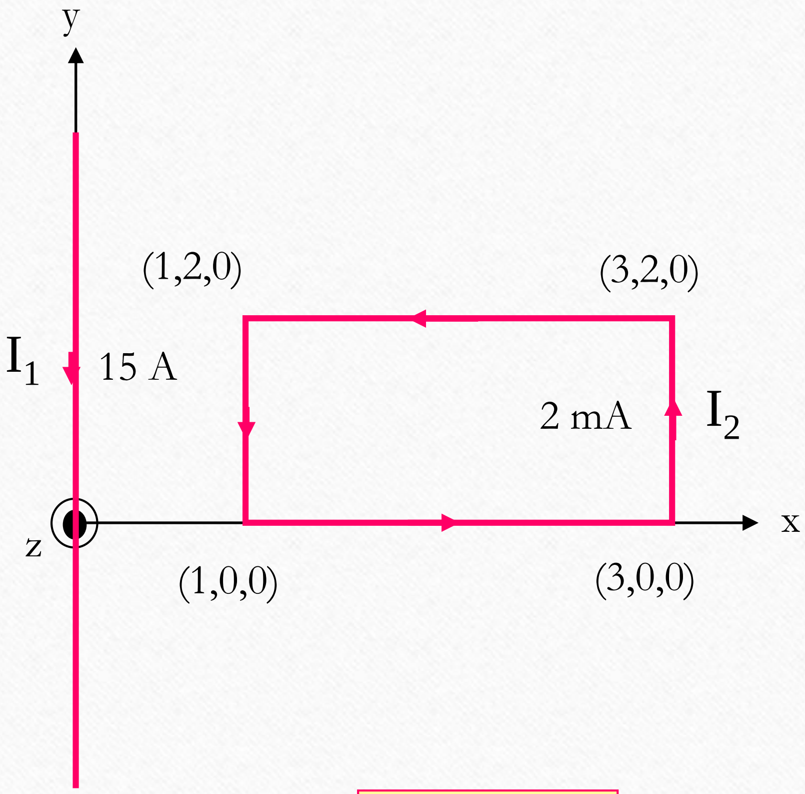
Force on a Differential Current Element

Solution:

Field created in the square loop due to filamentary current :

$$\overline{H}_1 = \frac{I_1}{2\pi x}\hat{z} = \frac{15}{2\pi x}\hat{z} \text{ A/m}$$

$$\overline{B}_1 = \mu_0 \overline{H}_1 = 4\pi \times 10^{-7}\,\overline{H}_1 = \frac{3\times 10^{-6}}{x}\hat{z} \text{ T}$$



$$\hat{\phi} = \hat{a}_l \times \hat{a}_R$$
$$= -\hat{y} \times \hat{x} = \hat{z}$$

Force on a Differential Current Element

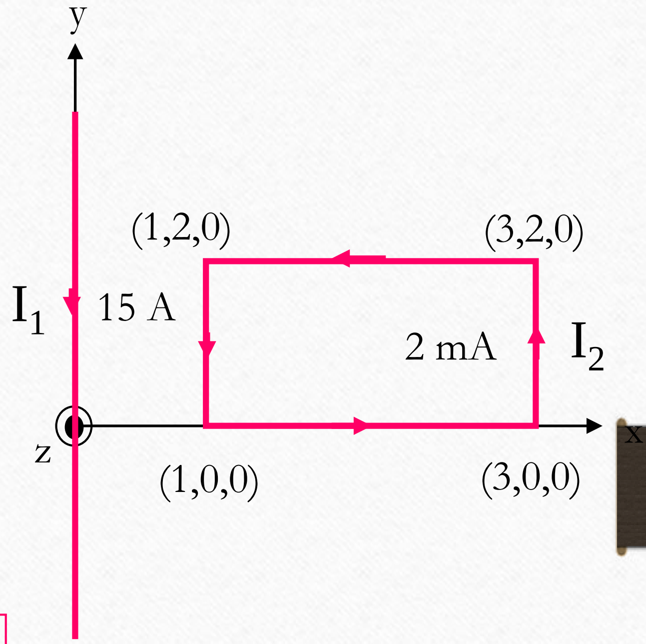
Hence: $\overline{F} = \oint I_2\, \overline{dl} \times \overline{B}_1 = -I_2 \oint \overline{B}_1 \times \overline{dl}$

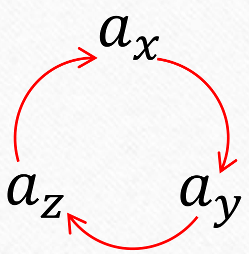
$$\overline{F} = -2\times10^{-3}\times3\times10^{-6}\left[\int_{x=1}^{3}\frac{\hat{z}}{x}\times dx\hat{x} + \int_{y=0}^{2}\frac{\hat{z}}{3}\times dy\hat{y} + \int_{x=3}^{1}\frac{\hat{z}}{x}\times dx\hat{x} + \int_{y=2}^{0}\frac{\hat{z}}{1}\times dy\hat{y}\right]$$

$$\overline{F} = -6\times10^{-9}\left[\ln x\Big|_1^3\hat{y} + \frac{1}{3}y\Big|_0^2(-\hat{x}) + \ln x\Big|_3^1\hat{y} + y\Big|_2^0(-\hat{x})\right]$$

$$= -6\times10^{-9}\left[(\ln 3)\hat{y} - \frac{2}{3}\hat{x} + \left(\ln\frac{1}{3}\right)\hat{y} + 2\hat{x}\right]$$

$$= -8\hat{x}\ \text{nN}$$



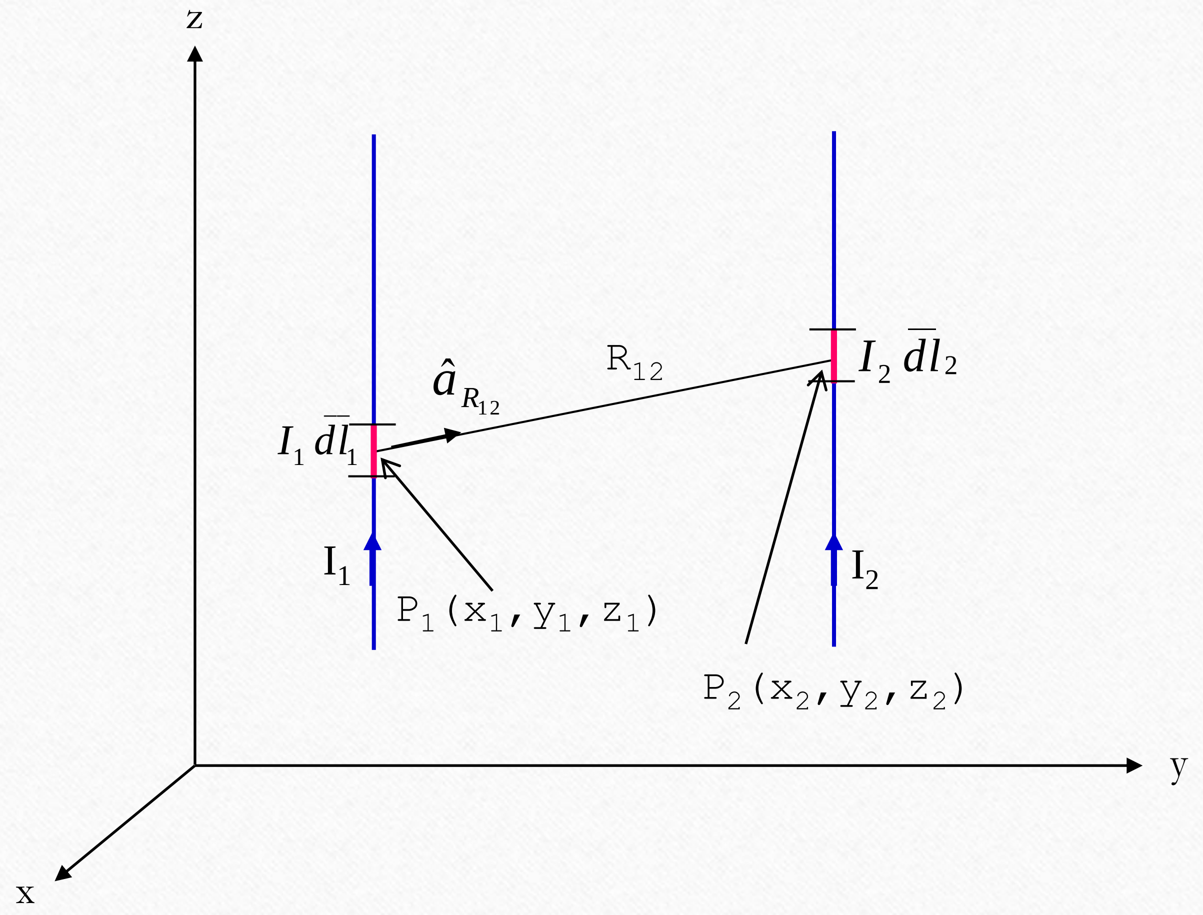


Force on a Differential Current Element

D8. 5. The field $\mathbf{B} = -2\mathbf{a}_x + 3\mathbf{a}_y + 4\mathbf{a}_z$ mT is present in free space. Find the vector force exerted on a straight wire carrying 12 A in the $\mathbf{a}_{AB}$ direction, given $A(1, 1, 1)$ and: (*a*) $B(2, 1, 1)$; (*b*) $B(3, 5, 6)$.

Ans. $-48\mathbf{a}_y + 36\mathbf{a}_z$ mN; $12\mathbf{a}_x - 216\mathbf{a}_y + 168\mathbf{a}_z$ mN

Force Between Differential Current Elements



Force Between Differential Current Elements

We have :

$$d\overline{F} = I d\overline{l} \text{ x } \overline{B} \quad \text{(N)}$$

The magnetic field at point P_2 due to the filamentary current $I_1 dl_1$:

$$d\overline{H}_2 = \frac{I_1 d\overline{l}_1 \text{ x } \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}} \quad \text{(A/m)}$$

$$d\left(d\overline{F}_2\right) = I_2 d\overline{l}_2 \text{ x } \frac{\mu_o I_1 d\overline{l}_1 \text{ x } \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}}$$

$$\left(d\overline{F}_2\right) = I_2 d\overline{l}_2 \text{ x } \oint_{l_1} \frac{\mu_o I_1 d\overline{l}_1 \text{ x } \hat{a}_{R_{12}}}{4\pi R_{12}^{\;2}} = I_2 d\overline{l}_2 \text{ x } \overline{B}_2$$

where $d\overline{F}_2$ is the force due to $I_2 dl_2$ and due to the magnetic field of wire l

Force Between Differential Current Elements

Integrate:

$$\bar{F}_2 = \oint_{l_2} I_2 dl_2 \text{ x } \oint_{l_1} \left[\frac{\mu_o I_1 d\bar{l}_1 \text{ x } \hat{a}_{R_{12}}}{4\pi R_{12}{}^2} \right]$$

$$\bar{F}_2 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{l_2} \left[\oint_{l_1} \frac{\left(\hat{a}_{R_{12}} \text{ x } d\bar{l}_1\right)}{R_{12}{}^2} \right] \text{ x } d\bar{l}_2$$

For surface current :

$$\bar{F}_2 = \int_s \bar{J}_{s2} \times \bar{B}_2 ds$$

For volume current :

$$\bar{F}_2 = \int_v \bar{J}_2 \times \bar{B}_2 dv$$

Force Between Differential Current Elements

D8. 6. Two differential current elements, $I_1\Delta\mathbf{L}_1 = 3 \times 10^{-6}\mathbf{a}_y$ A · m at $P_1(1, 0, 0)$ and $I_2\Delta\mathbf{L}_2 = 3 \times 10^{-6}(-0.5\mathbf{a}_x + 0.4\mathbf{a}_y + 0.3\mathbf{a}_z)$ A · m at $P_2(2, 2, 2)$, are located in free space. Find the vector force exerted on: (*a*) $I_2\Delta\mathbf{L}_2$ by $I_1\Delta\mathbf{L}_1$; (*b*) $I_1\Delta\mathbf{L}_1$ by $I_2\Delta\mathbf{L}_2$.

Ans. $(-1.333\mathbf{a}_x + 0.333\mathbf{a}_y - 2.67\mathbf{a}_z)10^{-20}$ N; $(4.67\mathbf{a}_x + 0.667\mathbf{a}_z)10^{-20}$ N

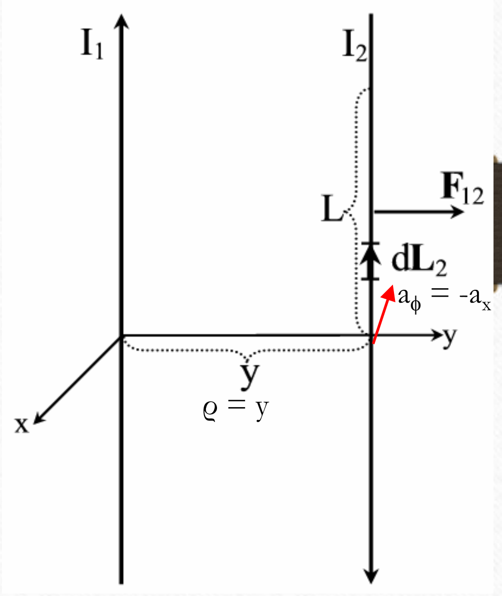
Magnetic Force between Two current Elements

Now let us consider a second line of current parallel to the first.
The force $d\mathbf{F}_{12}$ from the magnetic field of line 1 acting on a differential section of line 2 is

$$d\mathbf{F}_{12} = I_2 d\mathbf{L}_2 \times \mathbf{B}_1$$

The magnetic flux density $\mathbf{B}_1$ for an infinite length line of current is recalled from equation to be

$$\mathbf{B}_1 = \frac{\mu_o I_1}{2\pi\rho}\mathbf{a}_\phi$$



By inspection of the figure we see that ρ = y and $\mathbf{a}_\phi = -\mathbf{a}_x$. Inserting this in the above equation and considering that $d\mathbf{L}_2 = dz\mathbf{a}_z$, we have

$$\mathbf{F}_{12} = \int I_2 d\mathbf{L}_2 \times \mathbf{B}_1 = \int I_2 dz\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho}\mathbf{a}_\phi = \int I_2 dz\mathbf{a}_z \times \frac{\mu_o I_1}{2\pi\rho} - \mathbf{a}_x$$

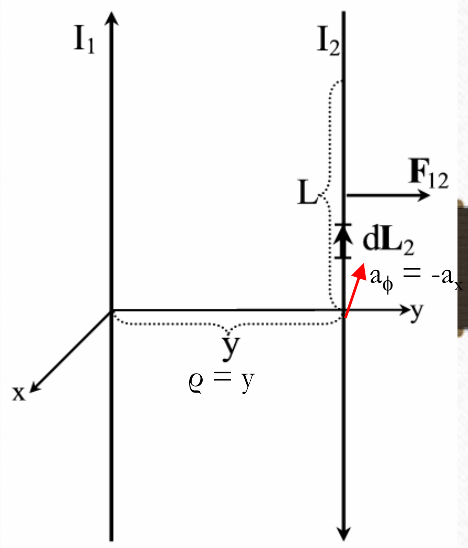
$$\mathbf{F}_{12} = \frac{\mu_o I_1 I_2}{2\pi y}\left(-\mathbf{a}_y\right)\int dz$$

Magnetic Force between Two current Elements

To find the total force on a length L of line 2 from the field of line 1, we must integrate d$\mathbf{F}_{12}$ from +L to 0. We are integrating in this direction to account for the direction of the current.

$$\mathbf{F}_{12} = \frac{\mu_o I_1 I_2}{2\pi y}\left(-\mathbf{a}_y\right)\int_L^0 dz$$

$$= \frac{\mu_o I_1 I_2 L}{2\pi y}\mathbf{a}_y$$



This gives us a repulsive force.

Had we instead been seeking $\mathbf{F}_{21}$, the magnetic force acting on line 1 from the field of line 2, we would have found $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

Conclusion:

1) Two parallel lines with current in opposite directions experience a force of repulsion.

2) For a pair of parallel lines with current in the same direction, a force of attraction would result.

Magnetic Force between Two current Elements

In the more general case where the two lines are not parallel, or not straight, we could use the Law of Biot-Savart to find $\mathbf{B}_1$ and arrive at

$$F_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \iint \frac{dL_2 \times (dL_1 \times a_{12})}{R_{12}^2}$$

This equation is known as *Ampere's Law of Force* between a pair of current carrying circuits and is analogous to Coulomb's law of force between a pair of charges.

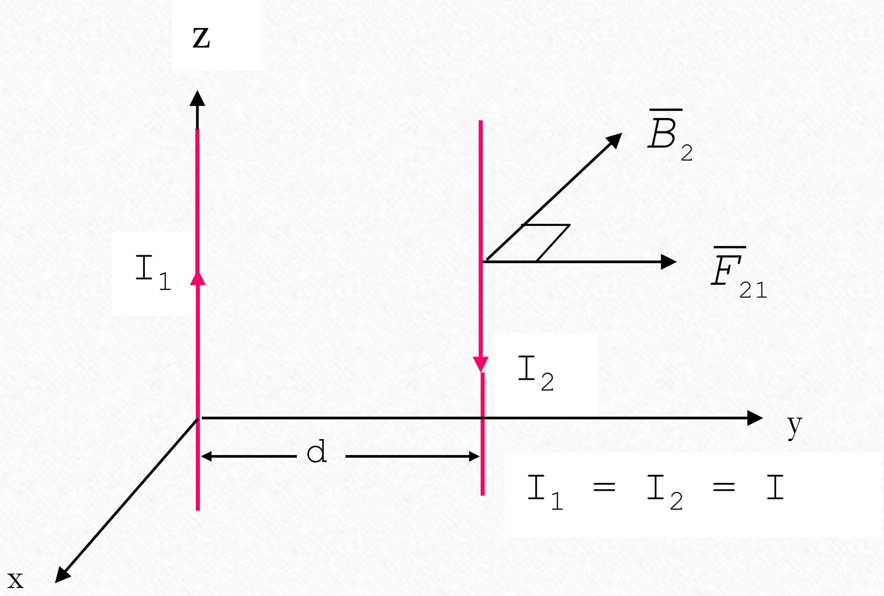
Magnetic Force between Two current Elements

Example: Find force per meter between two parallel infinite conductor carrying current, I Ampere in opposite direction and separated at a distance d meter.

Solution:

$\overline{B}_2$ at position conductor 2

$$\overline{B}_2 = \mu_0 \overline{H}_2 = \mu_0 \frac{\hat{\phi} I_1}{2\pi r_c} = \frac{-\hat{x}\mu_0 I_1}{2\pi d}$$



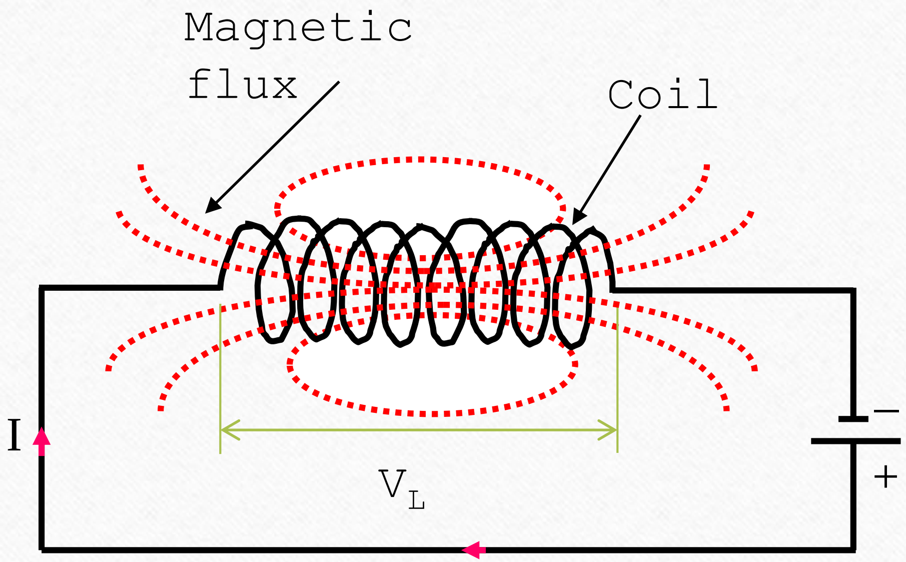
Hence:

$$\overline{F}_2 = \int_0^1 I_2 d\overline{\ell}_2 \times \left(\frac{-\hat{x}\mu_0 I_1}{2\pi d}\right) = \int_0^1 I_2(-\hat{z}dz) \times \left(\frac{-\hat{x}\mu_0 I_1}{2\pi d}\right)$$

$$= \hat{y}\mu_0 \frac{I_1 I_2}{2\pi d} = \hat{y}\frac{\mu_0 I^2}{2\pi d} \qquad \text{(N/m)}$$

Self Inductance

Inductance is the last of the three familiar parameters from circuit theory that we are defining in more general terms.



We can define the inductance (or self-inductance) as the ratio of the total flux linkages to the current which they link,

$$L = \frac{\Lambda}{I} = \frac{N\Phi}{I} \quad \text{Henry}$$

where Λ (lambda) is the total flux linkage of the inductor

From circuit theory the induced potential across a wire wound coil such as solenoid or a toroid :

$$V_L = L\frac{dI}{dt}$$

where L is the inductance of the coil, I is the time varying current flowing through the coil – inductor.

Self Inductance

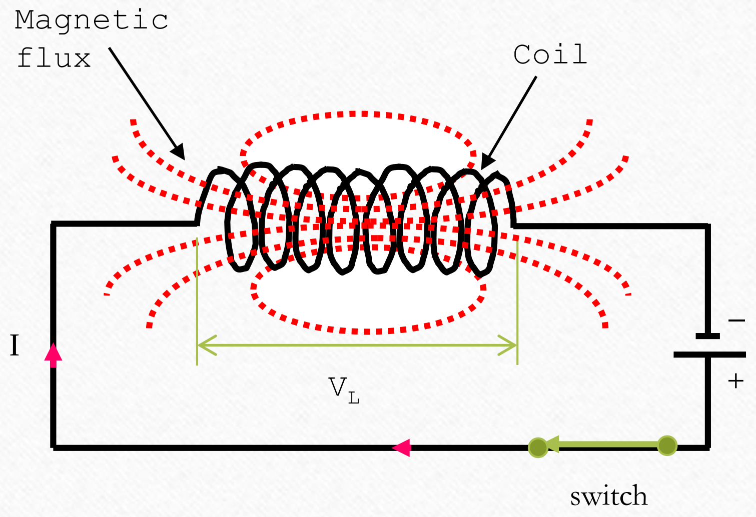
In a capacitor, the energy is stored in the electric field :

$$W_E = \frac{1}{2}CV^2$$

In an inductor, the energy is stored in the magnetic field, as suggested in the diagram :

$$W_m = \int_{t=0}^{t=t_0} V_L I dt = \int_{t=0}^{t=t_0} \left(L\frac{dI}{dt} \right) I dt$$

$$= \int_{t=0}^{t=t_0} LI\, dI = \frac{1}{2}LI^2 \quad (Joule)$$



Self Inductance

Example: Obtain the expression for self inductance per meter of the coaxial cable when the current flow is restricted to the surface of the inner conductor and the inner surface of the outer conductor as shown in the diagram.

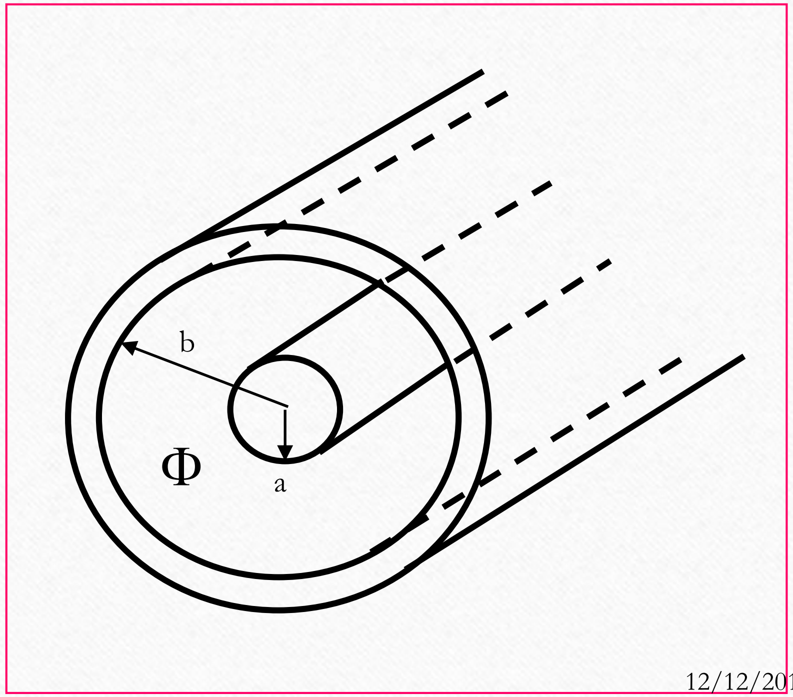
Solution:

The Φ will exist only between a and b and will link all the current I

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \int_0^1 \int_a^b \frac{(\mu H)(dr_c dz)}{I}$$

$$= \int_0^1 \int_a^b \frac{\mu I}{2\pi r_c} \frac{(dr_c dz)}{I}$$

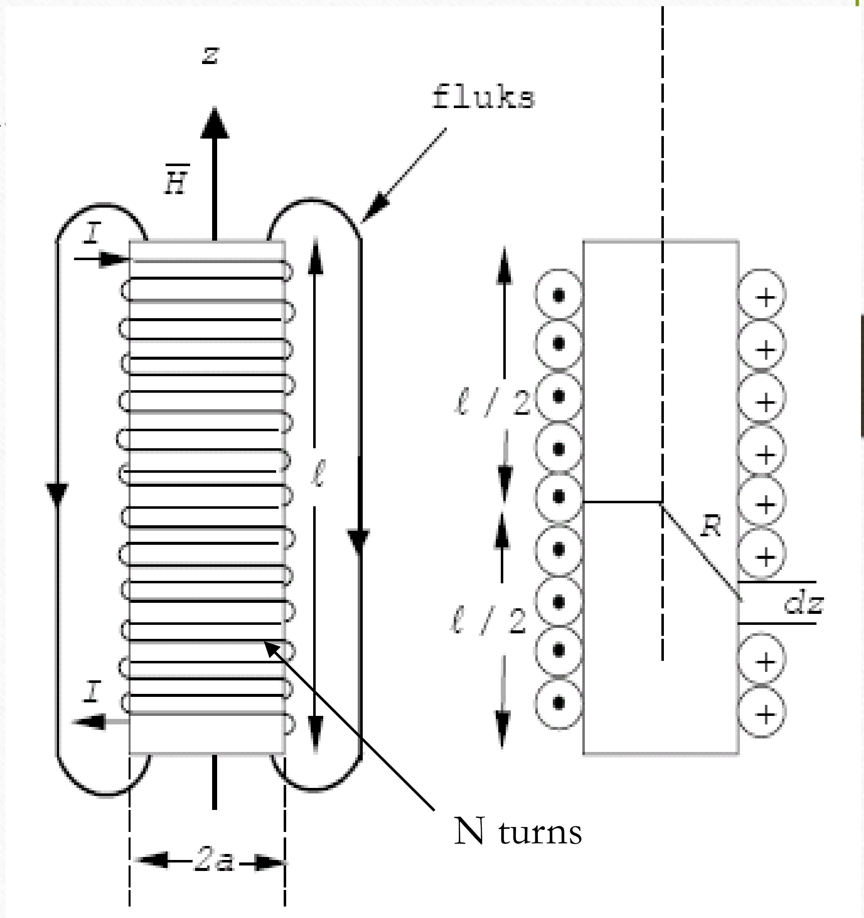
$$= \frac{\mu}{2\pi} \ln \frac{b}{a}$$



Self Inductance

Example: Obtain the self inductance of the long solenoid shown in the diagram.

Solution: Assume all the flux Φ links all N turns and that $\overline{B}$ does not vary over the cross section area of the solenoid.



$$\Lambda = \Phi N = B\left(\pi a^2\right)N$$

We have $\overline{B} = \mu \overline{H}$

$$\Lambda = (\mu H)\left(\pi a^2\right)N = \left(\frac{\mu N I}{l}\right)\left(\pi a^2\right)N$$

$$= \left(\frac{\mu N^2 I}{l}\right)\left(\pi a^2\right)$$

$$\therefore L = \frac{\Lambda}{I} = \frac{\mu N^2 \pi a^2}{l}$$

Self Inductance

Example: Obtain the self inductance of the toroid shown in the diagram.

Solution:

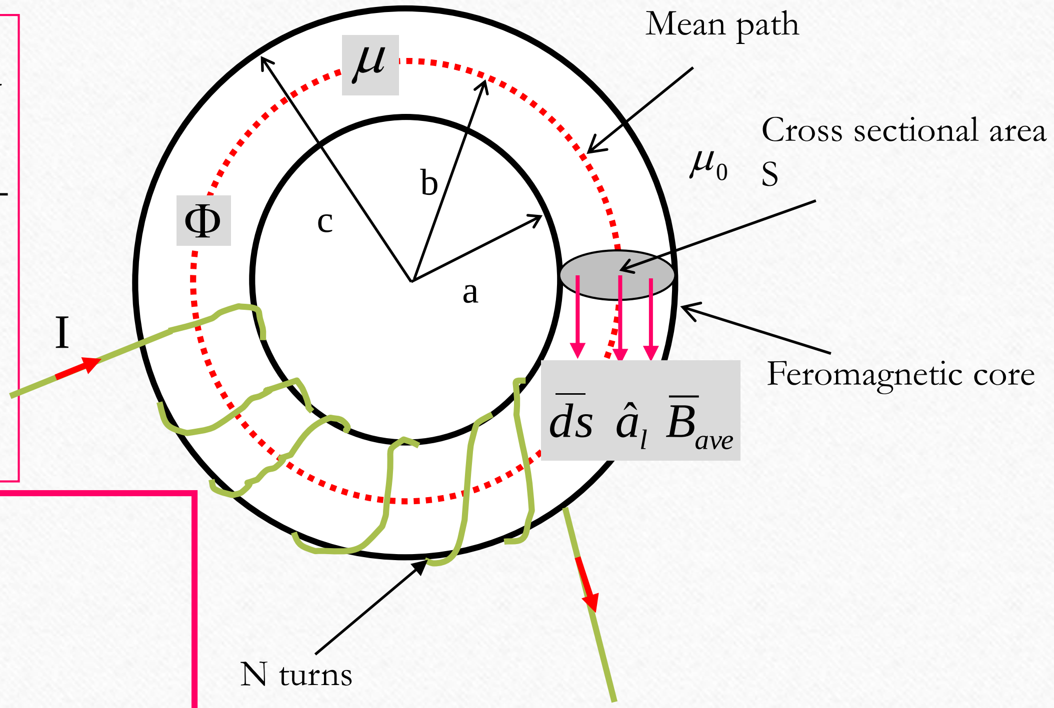
$$L = \frac{\Lambda}{I} = \frac{\Phi N}{I} = \frac{B\left(\frac{\pi(c-a)^2}{4}\right)N}{I}$$

$$= \frac{\mu \frac{NI}{2\pi b} SN}{I}$$

$$\therefore L = \frac{\mu N^2 S}{2\pi b}$$

where b - mean radius

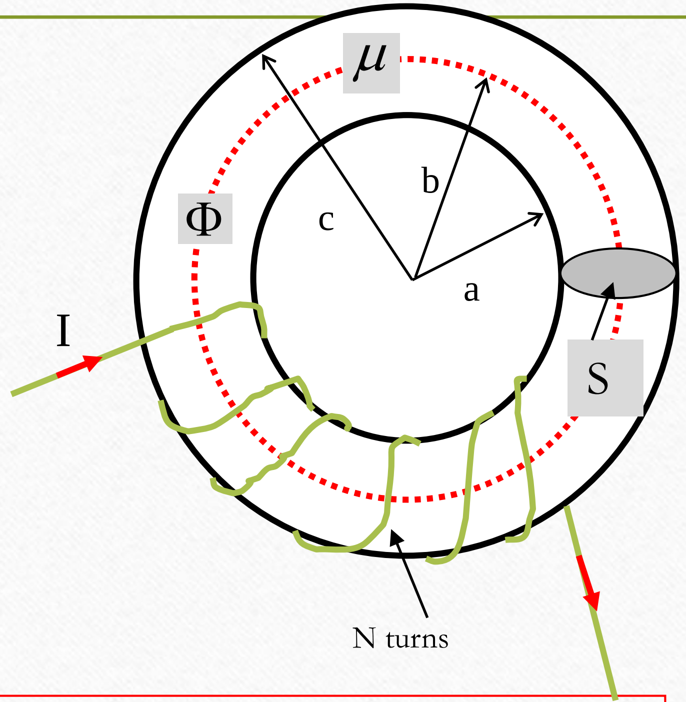
S - toroidal cross sectional area



Magnetic Energy Density

We have : $L \cong \frac{\Lambda}{I}$ Henry

$$W_m = \frac{1}{2}LI^2 = \frac{1}{2}\frac{\Lambda}{I}I^2 = \frac{1}{2}\Lambda I \quad \text{Joule}$$



Consider a toroidal ring : The energy in the magnetic field : $W_m = \frac{1}{2}\Phi NI = \frac{1}{2}BSNI$

Multiplying the numerator and denominator by 2πb : $W_m = \frac{1}{2}B\frac{NI}{2\pi b}(S2\pi b)$

where $\frac{NI}{2\pi b} = H$ and $(S2\pi b)$ is the volume V of the toroid

Magnetic Energy Density

Hence : $$W_m = \frac{1}{2} BHV$$

$$w_m = \frac{W_m}{V} = \frac{1}{2} BH = \frac{1}{2} \mu H^2 \quad Jm^{-3}$$

In vector form : $$w_m = \frac{1}{2} \overline{B} \cdot \overline{H}$$

Magnetic Energy Density

Example: Derive the expression for stored magnetic energy in a coaxial cable with the length l and the radius of the inner conductor a and the inner radius of the outer conductor is b. The permeability of the dielectric is μ .

Solution:

$$H = \frac{I}{2\pi r}$$

$$W_m = \frac{1}{2}\int_v \mu H^2 dv = \frac{\mu I^2}{8\pi^2}\int_v \frac{1}{r^2} dv$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l\, dr$$

$$= \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) \quad \text{(J)}$$